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Casimir surface forces on dielectric media in spherical geometry

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Abstract. This paper contains a calculation of the Casimir surface force density in spherical geometry under three different circumstances: (i) The system is an infinitely thin, perfectly conducting shell, endowed with dispersive properties. The presence of dispersion means that the earlier expressions calculated by Boyer (1968) and others have to be generalized; in particular, it is possible to revisit the old semiclassical electron idea of Casimir (1956). (ii) The system consists of two different spherical shells, of the same type as above. In particular, the non-dispersive Casimir surface force between two flat plates is recovered as the leading term in the formalism when the curvatures of the shells go to zero. (iii) The system is a compact dielectric ball, surrounded by a vacuum.

General formulae are given in all three cases, consistency checks carried out, and some simplifying approximations are given. All physical expressions, if necessary regularized by the Riemann zeta function method, are clear-cut and finite.

1. Introduction

Consider the stationary quantum electromagnetic zero-point fluctuations in a spherically symmetric dielectric system, typically a dielectric ball of radius a, surrounded by a vacuum. The task to be considered in this paper is to calculate the Casimir surface force density F on the system, using the Maxwell stress tensor. Let $S_{\mu\nu}$ be the Minkowski energy–momentum tensor for the electromagnetic field. The four-force density f_μ in the system is thus $f_\mu = -\partial_\nu S_{\mu\nu}$, and is, for a homogeneous sphere, different from zero only in the boundary layer around $r = a$. The time derivative of the electromagnetic momentum density does not play any role under stationary conditions, as assumed here, and the spatial components of the four-force density can accordingly be written as $f_i = -\partial_k S_{ik}$, where the sum over k runs from 1 to 3. (The Maxwell stress tensor is equal to $-S_{ik}$.) Integrating the radial component f_r across the boundary layer, we obtain in obvious notation

$$F = S_{rr}(a-) - S_{rr}(a+)\,. \tag{1}$$

We shall assume that the medium is non-magnetic, $\mu = 1$, and that it is dispersive with a frequency-dependent permittivity $\varepsilon(\omega)$. We may thus write the linear constituent relation for the material as $\boldsymbol{D} = \hat{\varepsilon}\boldsymbol{E}$, where $\hat{\varepsilon}$ is an integral operator defined such that the constitutive relation reads $\boldsymbol{D}(\omega) = \varepsilon(\omega)\boldsymbol{E}(\omega)$ in Fourier space (see, for instance, [1] p 77). In classical electromagnetic theory we have

$$S_{rr} = -\tfrac{1}{2}\hat{\varepsilon}\big(E_r^2 - E_\perp^2\big) - \tfrac{1}{2}\big(H_r^2 - H_\perp^2\big) \tag{2}$$

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where $E_\perp$ denotes the component of E transverse to the radius vector r.

In the quantal theory we consider the Green function $\Gamma(x, x')$ for two spacetime points x and x', and its Fourier transform $\Gamma(r, r', \omega)$, defined such that

$$\Gamma(x, x') = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega\tau} \Gamma(r, r', \omega) \tag{3}$$

with $\tau = t - t'$. In non-dispersive theory, τ plays the role of a high-frequency cut-off parameter. From Maxwell's equations one derives the governing equation for Γ

$$-\nabla \times \nabla \times \Gamma(r, r', \omega) + \varepsilon(\omega)\, \omega^2 \Gamma(r, r', \omega) = -\omega^2 \mathbf{1} \delta(r - r') \tag{4}$$

and the effective product of two electric field components becomes

$$i\, \langle E_i(r)\, E_k(r') \rangle_\omega = \Gamma_{ik}(r, r', \omega)\,. \tag{5}$$

The solution of (4) leads to two scalar Green functions, $F_l(r, r')$ and $G_l(r, r')$. These must be constructed such that the electromagnetic boundary conditions at dielectric surfaces are satisfied, and also such that the basic requirements about finiteness at the origin and outgoing wave conditions at infinity are met. The two-point functions for the electric and magnetic fields in Fourier space are

$$i\langle E_r(r) E_r(r') \rangle_\omega = \frac{1}{\varepsilon(\omega) r r'} \sum_{\ell=1}^{\infty} \frac{2\ell + 1}{4\pi} \ell(\ell + 1) G_\ell(r, r') \tag{6}$$

$$i\langle E_\perp(r) E_\perp(r') \rangle_\omega = \sum_{\ell=1}^{\infty} \frac{2\ell + 1}{4\pi} \left[\omega^2 F_\ell(r, r') + \frac{1}{\varepsilon(\omega) r r'} \frac{\partial}{\partial r} r \frac{\partial}{\partial r'} r' G_\ell(r, r') \right] \tag{7}$$

$$i\langle H_r(r) H_r(r') \rangle_\omega = \frac{1}{r r'} \sum_{\ell=1}^{\infty} \frac{2\ell + 1}{4\pi} \ell(\ell + 1) F_\ell(r, r') \tag{8}$$

$$i\langle H_\perp(r) H_\perp(r') \rangle_\omega = \sum_{\ell=1}^{\infty} \frac{2\ell + 1}{4\pi} \left[\varepsilon(\omega) \omega^2 G_\ell(r, r') + \frac{1}{r r'} \frac{\partial}{\partial r} r \frac{\partial}{\partial r'} r' F_\ell(r, r') \right]. \tag{9}$$

Here, we assume the two points r and r' lie in the same angular direction. The radial difference $r - r'$, however, does not necessarily have to be small in (6)–(9).

The calculation of the quantal surface force in the dielectric ball problem may thus appear to be simple, at least in principle: we may first solve the governing equation (4), then calculate the two-point functions from (6)–(9) in the limit $r' \to r$, and finally find F from (2) and (1). However, experience has shown that the task is more complicated than one might expect beforehand. It turns out that the problems met are essentially of two types: (i) In the non-dispersive theory, in the absence of a cut-off parameter, there occur divergences at *high frequencies*. As already mentioned, one can avoid these divergences by introducing a time splitting τ serving as a cut-off [2, 3]. The question arises: is the high-frequency divergence a fictitious phenomenon, without physical significance? It might be tempting simply to count the divergent term as insignificant; however, it has turned out in later years that this term is a substitute for a quite real effect connected with the *dispersion* of the material. Candelas [4] seems to have been the first to emphasize the importance of dispersion in connection with the Casimir effect. His general arguments based upon quantum field theory, were later essentially supported by explicit model calculations for a relativistic medium, i.e. a medium satisfying $\varepsilon\mu = 1$ [5, 6]. In fact, inclusion of the dispersive effect may reverse the direction of the surface force. In the following we shall proceed so as to avoid the time splitting τ completely, and instead work with a simple dispersive relation for the material. This appears to be the most reasonable physical approach. We emphasize that we still require the two spacetime points to be *spatially* separated, $r - r'$ being small but always different from zero. (We may also mention that our choice of separating the two points in the radial direction is a very natural way of proceeding but nevertheless not always followed in quantum field theory. For instance, although in a somewhat different context, one may consult [7] for a treatment involving the separation of points in the azimutal, instead of in the radial, direction.) (ii) The second type of problem is that divergences occur in the formalism when the angular momentum variable ℓ is summed up to infinity. This phenomenon is characteristic for curved boundaries and are absent if the boundaries are plane. The angular momentum divergence is more difficult to deal with than the previous high-frequency divergence. Even use of the asymptotic Debye expansions for the Riccati–Bessel functions—these expansions are most accurate precisely in the region of high ℓ—turns out to lead to divergences when $\ell \to \infty$. As we shall show below, it is, however, possible to regularize these divergences in a consistent way by means of the Riemann zeta function. We obtain clear-cut answers for all physical surface forces.

We shall follow the strategy of approaching the complicated Casimir problem for dielectrics in successive steps, dealing with simpler situations first. In the next section we consider a single, perfectly conducting shell, with vacuum regions on the inside and on the outside. That is, we return to the situation considered by Boyer [8], Milton *et al* [3], and others. The essential new element in our analysis, as compared to the previous ones, is that we take dispersion into account, and thereby demonstrate explicitly the attractive part of the surface force arising from the absorption frequency, called ω_0. Moreover, we revive an old idea put forward by Casimir [9], according to which a semiclassical 'electron' is pictured as a perfectly conducting shell, and calculate the value of $x_0 \equiv \omega_0 a$ resulting from the requirement that the electromagnetic zero-point fluctuations stabilize the electron against Coulomb repulsion. Our calculation yields $x_0 = 0.397$.

As the next step in complexity we consider in section 3 a double singular shell, consisting of perfectly conducting surfaces at $r = a$ and $r = b$. As far as we know, this system has not been considered before. We give the Green functions in the region $a < r < b$, calculate the Casimir surface forces, and verify, in particular, that the non-dispersive surface force density reduces in the limit $a \to \infty$ to the expression $\pi^2/240d^4$, with $d = b - a$. This is the standard expression for the force between two plates.

In section 4 we finally turn to the compact dielectric ball, calculate the two-point functions on the inside and on the outside, and also the corresponding inside and outside surface force densities, F_{int} and F_{ext}, respectively. Making use of Riemann zeta-function regularization, the non-dispersive parts of F_{int} and F_{ext} are worked out in detail in the limiting cases of high-permittivity media ($n \gg 1$), and dilute media ($n \simeq 1$), n meaning the refractive index. Finally, as an illustration of the close connection between angular momentum divergence and curvature of the dielectric boundary, we consider in the appendix the planar one-surface geometry, and verify explicitly the absence of the divergence in that case.

We employ Heaviside–Lorentz units, and put $\hbar$ and c equal to unity.

Finally it may be worthwhile to give some further references to works on the Casimir effect. General reviews are given by Plunien *et al* [10] and by Barash and Ginzburg [11]. Other useful sources are the books by Lifshitz and Pitaevskii [12], Ginzburg [13], and the Casimir honorary volume [14]. Popular accounts of the effect are given by Power [15], Belinfante [16] and by Elizalde and Romeo [17].

2. Single spherical shell

We recall the physical situation: the radius of the shell is $r = a$, there is vacuum on the inside as well as on the outside, and the temperature is zero. The dispersion of the material will be accounted for in a very simple way: we shall in the general case take the permittivity $\varepsilon(\mathrm{i}\hat{\omega})$ as a function of the frequency $\hat{\omega}$ along the imaginary axis to be a step function, as illustrated in figure 1. The 'absorption' frequency ω_0 serves in the mathematical sense as a high-frequency cut-off; the dielectric properties of the material are absent for frequencies $\hat{\omega} > \omega_0$. Thus, in the case of perfect conductivity, $\varepsilon \to \infty$ for $\hat{\omega} \leqslant \omega_0$, while $\varepsilon = 1$ for $\hat{\omega} > \omega_0$.

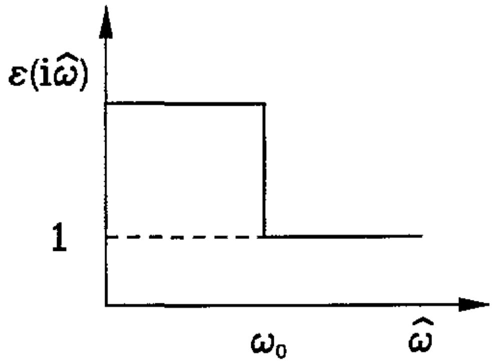

Figure 1. The adopted dispersion relation: permittivity ε versus frequency $\hat{\omega}$ along the imaginary frequency axis.

It ought to be emphasized that $\varepsilon(\mathrm{i}\hat{\omega})$ is required to have a step-function form along the *imaginary* axis only; the variation of $\varepsilon(\omega)$ along the real axis is not given and must be expected to be quite different. The value of $\varepsilon(\omega)$ along the real axis will have to adjust itself such that the dispersion relations of the Kramers–Kronig type, expressing the analytic properties of $\varepsilon(\omega)$ in the upper half of the complex frequency plane, are satisfied.

The two scalar Green functions $F_l(r, r')$ and $G_l(r, r')$ for the spherical shell are the following [3]. For $r, r' < a$,

$$\left\{ \begin{array}{c} F_\ell \\ G_\ell \end{array} \right\} = \mathrm{i}k\, j_\ell(kr_<)\left[h_\ell^{(1)}(kr_>) - \left\{ \begin{array}{c} [\tilde{e}_\ell(ka)/\tilde{s}_\ell(ka)] \\ [\tilde{e}'_\ell(ka)/\tilde{s}'_\ell(ka)] \end{array} \right\} j_\ell(kr_>)\right] \tag{10}$$

whereas for $r, r' > a$,

$$\left\{ \begin{array}{c} F_\ell \\ G_\ell \end{array} \right\} = \mathrm{i}k\left[j_\ell(kr_<) - \left\{ \begin{array}{c} [\tilde{s}_\ell(ka)/\tilde{e}_\ell(ka)] \\ [\tilde{s}'_\ell(ka)/\tilde{e}'_\ell(ka)] \end{array} \right\} h_\ell^{(1)}(kr_<)\right] h_\ell^{(1)}(kr_>) . \tag{11}$$

Here, $k = |\omega|$, j_ℓ and $h_\ell^{(1)}$ are the spherical Bessel and Hankel functions, and $\tilde{s}_\ell$ and $\tilde{e}_\ell$ are *the Riccati–Bessel functions in conventional normalization* [18]

$$\tilde{s}_\ell(x) = x j_\ell(x) \qquad \tilde{e}_\ell(x) = x h_\ell^{(1)}(x) \tag{12}$$

corresponding to the Wronskian $W\{\tilde{s}_\ell, \tilde{e}_\ell\} = \mathrm{i}$. The two-point functions for the electric and magnetic fields are now calculable from (6)–(9), in the limit $r \to r'$. We show briefly the calculation for the radial electric field only: inserting (10) into (6) we obtain for two neighbouring points in the interior region

$$\langle E_r(r)E_r(r')\rangle_\omega|_{r'\to r} = \frac{1}{kr^4}\sum_{\ell=1}^{\infty}\frac{2\ell+1}{4\pi}\ell(\ell+1)\tilde{s}_\ell(kr)\left[\tilde{e}_\ell(kr) - \frac{\tilde{e}'_\ell(ka)}{\tilde{s}'_\ell(ka)}\tilde{s}_\ell(kr)\right] \tag{13}$$

and in configuration space, in analogy to (3) when $\tau = 0$,

$$\langle E_r(r)E_r(r')\rangle_{r'\to r} = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \langle E_r(r)E_r(r')\rangle_\omega \bigg|_{r'\to r}. \tag{14}$$

The integral over ω can be replaced by twice the integral from zero to infinity. We perform a complex frequency rotation

$$k \to \mathrm{i}\hat{k} = \mathrm{i}\hat{\omega} \qquad ka \to \mathrm{i}\hat{\omega}a \equiv \mathrm{i}x \tag{15}$$

and define new Riccati–Bessel functions s_ℓ, e_ℓ related to the conventional ones $\tilde{s}_\ell$, $\tilde{e}_\ell$ by

$$s_\ell(x) = (-\mathrm{i})^{\ell+1}\tilde{s}_\ell(\mathrm{i}x) = \sqrt{\frac{\pi x}{2}}\, I_\nu(x) \qquad e_\ell(x) = \mathrm{i}^{\ell+1}\tilde{e}_\ell(\mathrm{i}x) = \sqrt{\frac{2x}{\pi}}\, K_\nu(x) \tag{16}$$

where $\nu = \ell + \frac{1}{2}$. The new normalization corresponds to the Wronskian $W\{s_\ell, e_\ell\} = -1$. We get, in configuration space, for $r, r' < a$,

$$\langle E_r(r)E_r(r')\rangle_{r'\to r} = \frac{1}{\pi r^4}\int_0^{x_0} \frac{\mathrm{d}x}{x} \sum_{\ell=1}^{\infty} \frac{2\ell+1}{4\pi}\ell(\ell+1)\left[s_\ell\left(\frac{xr_<}{a}\right)e_\ell\left(\frac{xr_>}{a}\right) - \frac{e_\ell'(x)}{s_\ell'(x)}s_\ell^2\left(\frac{xr}{a}\right)\right]. \tag{17}$$

Here, the integration over $x = \hat{\omega}a$ has been terminated at

$$x_0 = \omega_0 a \tag{18}$$

in accordance with the dispersion relation shown in figure 1. One may note that there is no divergence at the lower limit of the integral in (17); this can be seen from the approximate forms [5]

$$s_\ell(x) = \frac{\sqrt{\pi}}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^{\ell+1} \qquad e_\ell(x) = \frac{\Gamma(\nu)}{\sqrt{\pi}}\left(\frac{x}{2}\right)^{-\ell} \tag{19}$$

which hold when $x \gg 1$.

A similar calculation for the other two-point functions yields, for $r, r' < a$,

$$\begin{aligned}\langle E_\perp(r)E_\perp(r')\rangle_{r'\to r} = \frac{-1}{\pi r^2 a^2}\int_0^{x_0} x\,\mathrm{d}x \sum_{\ell=1}^{\infty} \frac{2\ell+1}{4\pi}\bigg\{&s_\ell\left(\frac{xr_<}{a}\right)e_\ell\left(\frac{xr_>}{a}\right) - s_\ell'\left(\frac{xr_<}{a}\right)e_\ell'\left(\frac{xr_>}{a}\right)\\ &-\frac{e_\ell(x)}{s_\ell(x)}s_\ell^2\left(\frac{xr}{a}\right) + \frac{e_\ell'(x)}{s_\ell'(x)}\left[s_\ell'\left(\frac{xr}{a}\right)\right]^2\bigg\}\end{aligned} \tag{20}$$

$$\begin{aligned}\langle H_r(r)H_r(r')\rangle_{r'\to r} = \frac{1}{\pi r^4}\int_0^{x_0} \frac{\mathrm{d}x}{x} \sum_{\ell=1}^{\infty} \frac{2\ell+1}{4\pi}\ell(\ell+1)\\ \times\left[s_\ell\left(\frac{xr_<}{a}\right)e_\ell\left(\frac{xr_>}{a}\right) - \frac{e_\ell(x)}{s_\ell(x)}s_\ell^2\left(\frac{xr}{a}\right)\right]\end{aligned} \tag{21}$$

$$\begin{aligned}\langle H_\perp(r)H_\perp(r')\rangle_{r'\to r} = \frac{-1}{\pi r^2 a^2}\int_0^{x_0} x\,\mathrm{d}x \sum_{\ell=1}^{\infty} \frac{2\ell+1}{4\pi}\bigg\{&s_\ell\left(\frac{xr_<}{a}\right)e_\ell\left(\frac{xr_>}{a}\right) - s_\ell'\left(\frac{xr_<}{a}\right)e_\ell'\left(\frac{xr_>}{a}\right)\\ &-\frac{e_\ell'(x)}{s_\ell'(x)}s_\ell^2\left(\frac{xr}{a}\right) + \frac{e_\ell(x)}{s_\ell(x)}\left[s_\ell'\left(\frac{xr}{a}\right)\right]^2\bigg\}.\end{aligned} \tag{22}$$

On the outside, for $r, r' > a$ we have

$$\langle E_r(r)E_r(r')\rangle_{r'\to r} = \frac{1}{\pi r^4}\int_0^{x_0} \frac{\mathrm{d}x}{x} \sum_{\ell=1}^{\infty} \frac{2\ell+1}{4\pi}\ell(\ell+1)\left[s_\ell\left(\frac{xr_<}{a}\right)e_\ell\left(\frac{xr_>}{a}\right) - \frac{s_\ell'(x)}{e_\ell'(x)}e_\ell^2\left(\frac{xr}{a}\right)\right] \tag{23}$$

$$\langle E_\perp(r)E_\perp(r')\rangle_{r'\to r} = \frac{-1}{\pi r^2 a^2}\int_0^{x_0} x\,\mathrm{d}x \sum_{\ell=1}^{\infty}\frac{2\ell+1}{4\pi}\left\{s_\ell\left(\frac{xr_<}{a}\right)e_\ell\left(\frac{xr_>}{a}\right) - s'_\ell\left(\frac{xr_<}{a}\right)e'_\ell\left(\frac{xr_>}{a}\right)\right.$$
$$\left. -\frac{s_\ell(x)}{e_\ell(x)}e_\ell^2\left(\frac{xr}{a}\right) + \frac{s'_\ell(x)}{e'_\ell(x)}\left[e'_\ell\left(\frac{xr}{a}\right)\right]^2\right\} \tag{24}$$

$$\langle H_r(r)H_r(r')\rangle_{r'\to r} = \frac{1}{\pi r^4}\int_0^{x_0}\frac{\mathrm{d}x}{x}\sum_{\ell=1}^{\infty}\frac{2\ell+1}{4\pi}\ell(\ell+1)\left[s_\ell\left(\frac{xr_<}{a}\right)e_\ell\left(\frac{xr_>}{a}\right) - \frac{s_\ell(x)}{e_\ell(x)}e_\ell^2\left(\frac{xr}{a}\right)\right] \tag{25}$$

$$\langle H_\perp(r)H_\perp(r')\rangle_{r'\to r} = \frac{-1}{\pi r^2 a^2}\int_0^{x_0} x\,\mathrm{d}x \sum_{\ell=1}^{\infty}\frac{2\ell+1}{4\pi}\left\{s_\ell\left(\frac{xr_<}{a}\right)e_\ell\left(\frac{xr_>}{a}\right) - s'_\ell\left(\frac{xr_<}{a}\right)e'_\ell\left(\frac{xr_>}{a}\right)\right.$$
$$\left. -\frac{s'_\ell(x)}{e'_\ell(x)}e_\ell^2\left(\frac{xr}{a}\right) + \frac{s_\ell(x)}{e_\ell(x)}\left[e'_\ell\left(\frac{xr}{a}\right)\right]^2\right\}. \tag{26}$$

The surface force density on the shell can in accordance with (1) and (2) be written as the difference between the quantities $-\frac{1}{2}(E_r^2 - H_\perp^2)$ evaluated on the inside and on the outside:

$$F = -\tfrac{1}{2}\left[\langle E_r(r)E_r(r')\rangle - \langle H_\perp(r)H_\perp(r')\rangle\right]_{r'\to r=a+}^{r'\to r=a-}. \tag{27}$$

Here we have taken into account the relations

$$\langle E_\perp(r)E_\perp(r')\rangle_{r'\to r=a\pm} = \langle H_r(r)H_r(r')\rangle_{r'\to r=a\pm} = 0 \tag{28}$$

which express the boundary conditions at the surface (they follow also formally from (20), (21), (24), and (25)).

Substitution into (27) yields, when we take into account the differential equation for the frequency-rotated Riccati–Bessel functions, the surface force density

$$F = \frac{-1}{2\pi a^4}\int_0^{x_0} x\,\mathrm{d}x\sum_{\ell=1}^{\infty}\frac{2\ell+1}{4\pi}\left[\frac{s'_\ell(x)}{s_\ell(x)} + \frac{s''_\ell(x)}{s'_\ell(x)} + \frac{e'_\ell(x)}{e_\ell(x)} + \frac{e''_\ell(x)}{e'_\ell(x)}\right]. \tag{29}$$

Strictly speaking, equation (29) gives the non-regularized force. However, in the present case the non-regularized force becomes identical to the regularized one. This is so because the contact term is equal to the force calculated if the inside region, respectively the outside region, were uniform (i.e. without boundaries). These uniform regions correspond to the $ik\, j_\ell(kr_<)\, h_\ell^{(1)}(kr_>)$ terms in (10) and (11), and are thus equal to each other at the surface. This means that the contact term for the force vanishes, and (29) gives accordingly the physical force as it stands.

Equation (29) is in accordance with the result of Milton *et al* [3], except from the dispersion-induced upper limit x_0. We shall now show how the expression can be further processed in a fairly simple way, up to the leading term in accuracy, by making use of the Debye expansions for the Riccati–Bessel functions. First, rewrite (29) such that it contains the derivative of the logarithm of a product

$$F = \frac{-1}{2\pi a^4}\int_0^{x_0} x\,\mathrm{d}x\sum_{\ell=1}^{\infty}\frac{2\ell+1}{4\pi}\frac{\mathrm{d}}{\mathrm{d}x}\log(-s_\ell s'_\ell e_\ell e'_\ell) \tag{30}$$

and insert this into the Debye expansions [18, 19]

$$s_\ell(x) = \tfrac{1}{2}\, z^{1/2}\, t^{1/2}(z)\, \mathrm{e}^{\nu\eta(z)}\left[1 + \sum_{k=1}^{\infty}\frac{u_k(t)}{\nu^k}\right] \tag{31}$$

$$e_\ell(x) = z^{1/2}\, t^{1/2}(z)\, \mathrm{e}^{-\nu\eta(z)}\left[1 + \sum_{k=1}^{\infty}(-1)^k \frac{u_k(t)}{\nu^k}\right] \tag{32}$$

with similar expansions for $s'_\ell(x)$ and $e'_\ell(x)$. Here $z = x/\nu$, and

$$t(z) = \left(1+z^2\right)^{-1/2} \tag{33}$$

$$\eta(z) = \left(1+z^2\right)^{1/2} + \log \frac{z}{1+\left(1+z^2\right)^{1/2}}\,. \tag{34}$$

The expressions for the coefficients u_k, v_k will not be written down here; they are given for $k = 1, 2, 3$ in [18] and for values of k up to $k = 6$ in [19]. Sufficient in our context is the simple relationship

$$-s_\ell s'_\ell e_\ell e'_\ell = \frac{1}{4}\left[1 - \frac{t^6}{4\nu^2} + \mathcal{O}\left(\frac{1}{\nu^4}\right)\right] \tag{35}$$

which shows that the leading contribution to the derivative of the logarithm in (30) is through the $\mathcal{O}(1/\nu^2)$ term. Using results calculated earlier in related works [6, 20], we obtain

$$F = \frac{-3}{8\pi^2 a^4}\left[\frac{x_0}{6} + \frac{x_0}{48}\,\frac{7+9x_0^2/w^2}{(1+x_0^2/w^2)^2} - \tfrac{55}{32}\tan^{-1}\left(\frac{x_0}{w}\right)\right.$$
$$\left. + \int_0^{x_0/w} z^2 t^4\left(\frac{w}{2} + \frac{7}{3w}t^4 + \frac{2}{3w}t^6\right)\mathrm{d}z + \sum_{\ell=1}^{4}\int_0^{x_0/\nu} z^2 t^8\,\mathrm{d}z\right]. \tag{36}$$

Here, $w = \frac{11}{2}$, $t = t(z)$ as given by (33), and the Euler–Maclaurin formula has been used. The integrals in (36) can be evaluated and we obtain the following compact and explicit form:

$$F = \frac{-3}{8\pi^2 a^4}\left[\frac{w}{6}\mathcal{M}\left(\frac{x_0}{w}\right) + \frac{w}{2}\mathcal{N}_4\left(\frac{x_0}{w}\right) + \frac{7}{3w}\mathcal{N}_8\left(\frac{x_0}{w}\right) + \frac{2}{3w}\mathcal{N}_{10}\left(\frac{x_0}{w}\right) + \sum_{\ell=1}^{4}\mathcal{N}_8\left(\frac{x_0}{\nu}\right)\right] \tag{37}$$

where we have defined the following functions:

$$\begin{aligned}
\mathcal{M}(x) &\equiv \int_0^x z^6 t^6(z)\,\mathrm{d}z = x - \tfrac{15}{8}\tan^{-1}(x) + \frac{x(7+9x^2)}{8(1+x^2)^2} \\
\mathcal{N}_4(x) &\equiv \int_0^x z^2 t^4(z)\,\mathrm{d}z = \tfrac{1}{2}\tan^{-1}(x) - \frac{x}{2(1+x^2)} \\
\mathcal{N}_8(x) &\equiv \int_0^x z^2 t^8(z)\,\mathrm{d}z = \tfrac{1}{16}\tan^{-1}(x) - \frac{x(3-8x^2-3x^4)}{48(1+x^2)^3} \\
\mathcal{N}_{10}(x) &\equiv \int_0^x z^2 t^{10}(z)\,\mathrm{d}z = \tfrac{5}{128}\tan^{-1}(x) - \frac{x\left(15-73x^2-55x^4-15x^6\right)}{384(1+x^2)^4}\,.
\end{aligned} \tag{38}$$

It is of interest to approximate (37) by a simpler, asymptotic expression in the limit $x_0 \to \infty$, which corresponds to the medium being non-dispersive. It is seen that the divergence of (37) in this case is linear in x_0. We obtain

$$F_{\text{asymptotic}} = \frac{1}{4\pi a^4}\left(-\frac{x_0}{4\pi} + \frac{3}{64}\right) \qquad \text{for large } x_0\,. \tag{39}$$

The condition imposed on x_0 in order to make (39) a useful approximation is weaker than one might expect. As mentioned already in [20], numerical checks show that a value of x_0 as low as 1.4, implies that the error in (39) is less than 1%. For $x_0 = 1$ the error is larger, about 10%. Thus for $x_0 \lesssim 1$, (39) ceases to be useful.

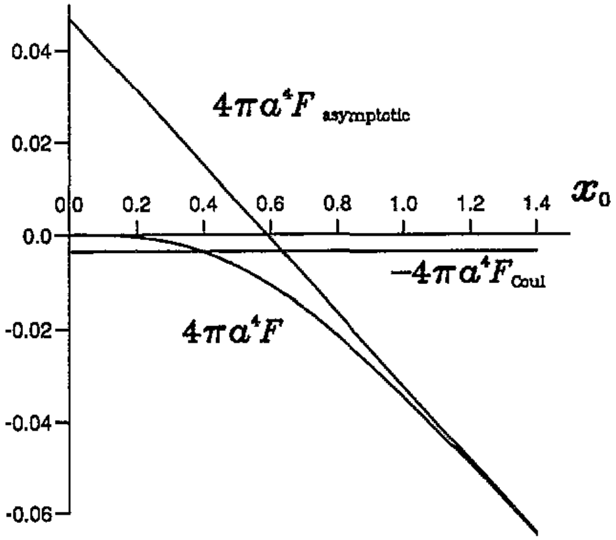

Figure 2. Showing the validity of the asymptotic force density expression (equation (39)).

The term $\frac{3}{64}$ in the parenthesis in (39) is characteristic for the non-dispersive theory. This is the accuracy following from our use of the Debye expansions up to the leading term. We have $\frac{3}{64} = \left(\frac{1}{2}\right) \times 0.093\,75$, which is to be compared with the number $\left(\frac{1}{2}\right) \times 0.092\,35$ calculated by Milton *et al* [3] in the more accurate non-dispersive calculation. The presence of the x_0 term is interesting from a physical viewpoint, since it means that there is an *attractive* component in the force, the strength of which is a function of the magnitude of x_0. It becomes natural here to recall the semiclassical electron model proposed a long time ago by Casimir [9]: the idea was to calculate the magnitude of the fine-structure constant $\alpha = e^2/4\pi$ by requiring balance between the inward-directed force arising from zero-point fluctuations and the outward-directed Coulomb force. The latter force is, per unit area,

$$F_{\rm Coul} = \frac{e^2}{32\pi^2 a^4} = \frac{\alpha}{8\pi a^4} \tag{40}$$

where e is the charge on the shell. Evidently we are unable to calculate the value of α here, but we can answer a more modest question: what is the magnitude of x_0 that follows from the mentioned requirement, $F + F_{\rm Coul} = 0$, if $\alpha = \frac{1}{137}$ is used as an input parameter? It is seen that this value of x_0 is a pure number, independent of the radius a. First using the simple equation (39) together with (40) we obtain $x_0 \simeq 0.63$, which, however, is too small a value to be regarded with confidence in relation to the approximate equation (39). This is readily seen from figure 2, where we have plotted the force for small values of x_0, both according to (37) and (39). A numerical calculation, based upon (37), yields

$$x_0 \equiv \omega_0 a = 0.397\,. \tag{41}$$

For example, put a equal to 2.8 fm, the classical electron radius. Then, equation (41) implies in dimensional units that the cut-off for the classical 'electron' equals $\omega_0 = x_0 c/a$ $= 4.3\times10^{22}\ {\rm s}^{-1}$.

Before leaving the theory of the single shell, we note the following property of the two-point functions given in (17), (20)–(26). These functions refer to the *total* electric or magnetic field components, in the inside or in the outside region. Thus the terms referring to the uniform vacuum are kept in the scalar Green functions F_ℓ and G_ℓ; cf the first terms on the right-hand sides in (10) and (11). Now, one may instead choose to work with the two-point functions that contain the surface-induced contributions only, and thus to subtract off the uniform vacuum terms from the beginning. This may be a simplifying way to proceed in some cases, in particular, because the two-point functions always refer to 'disturbed' quantities caused by the presence of the boundaries. There is thus no need for regularizing physical expressions later on. We shall call these new functions the 'effective' two-point functions. For reference purpose we write them down here, for the electric field components. For $r, r' < a$,

$$\langle E_r(r)E_r(r')\rangle_{\text{eff}} = \frac{-1}{\pi r^4}\int_0^{x_0}\frac{\mathrm{d}x}{x}\sum_{\ell=1}^{\infty}\frac{2\ell+1}{4\pi}\ell(\ell+1)\frac{e'_\ell(x)}{s'_\ell(x)}s_\ell^2\left(\frac{xr}{a}\right) \tag{42}$$

$$\langle E_\perp(r)E_\perp(r')\rangle_{\text{eff}} = \frac{1}{\pi r^2 a^2}\int_0^{x_0} x\,\mathrm{d}x\sum_{\ell=1}^{\infty}\frac{2\ell+1}{4\pi}\left\{\frac{e_\ell(x)}{s_\ell(x)}s_\ell^2\left(\frac{xr}{a}\right) - \frac{e'_\ell(x)}{s'_\ell(x)}\left[s'_\ell\left(\frac{xr}{a}\right)\right]^2\right\}. \tag{43}$$

Here, the limit $r' \to r$ is understood but not written out explicitly. For $r, r' > a$ we have

$$\langle E_r(r)E_r(r')\rangle_{\text{eff}} = \frac{-1}{\pi r^4}\int_0^{x_0}\frac{\mathrm{d}x}{x}\sum_{\ell=1}^{\infty}\frac{2\ell+1}{4\pi}\ell(\ell+1)\frac{s'_\ell(x)}{e'_\ell(x)}e_\ell^2\left(\frac{xr}{a}\right) \tag{44}$$

$$\langle E_\perp(r)E_\perp(r')\rangle_{\text{eff}} = \frac{1}{\pi r^2 a^2}\int_0^{x_0} x\,\mathrm{d}x\sum_{\ell=1}^{\infty}\frac{2\ell+1}{4\pi}\left\{\frac{s_\ell(x)}{e_\ell(x)}e_\ell^2\left(\frac{xr}{a}\right) - \frac{s'_\ell(x)}{e'_\ell(x)}\left[e'_\ell\left(\frac{xr}{a}\right)\right]^2\right\}. \tag{45}$$

The effective two-point functions can evidently be used to calculate the surface force. But it is then necessary to start from the full stress tensor expression

$$F = -\tfrac{1}{2}\left[\langle E_r(r)E_r(r')\rangle - \langle E_\perp(r)E_\perp(r') + \langle H_r(r)H_r(r')\rangle - \langle H_\perp(r)H_\perp(r')\rangle\right]_{r'\to r=a+}^{r'\to r=a-} \tag{46}$$

instead of from the simplified expression (27). This is so because (27) is based upon (28), which require the fields to be the total fields. The analogous two-point functions for the effective fields do not vanish on the surface.

3. Double spherical shell

The situation is sketched in figure 3: there are two perfectly conducting singular shells situated at $r = a$ and $r = b$, with identical materials in the shells so that the 'absorption' frequencies are identical, equal to ω_0. In the inner region I, the annular region II, and the outer region III, we assume there to be a vacuum.

Let us first consider the two scalar Green functions, $F_\ell(r, r')$ and $G_\ell(r, r')$. In region I, they are given by (10), as before. In region II, they are

$$F_\ell(r, r') = \frac{\mathrm{i}k}{1 - \frac{\tilde{s}_\ell(1)\tilde{e}_\ell(2)}{\tilde{e}_\ell(1)\tilde{s}_\ell(2)}}\left[j_\ell(kr_<) - \frac{\tilde{s}_\ell(1)}{\tilde{e}_\ell(1)}\,h_\ell^{(1)}(kr_<)\right]\left[h_\ell^{(1)}(kr_>) - \frac{\tilde{e}_\ell(2)}{\tilde{s}_\ell(2)}\,j_\ell(kr_>)\right] \tag{47}$$

$$G_\ell(r, r') = \frac{\mathrm{i}k}{1 - \frac{\tilde{s}'_\ell(1)\tilde{e}'_\ell(2)}{\tilde{e}'_\ell(1)\tilde{s}'_\ell(2)}}\left[j_\ell(kr_<) - \frac{\tilde{s}'_\ell(1)}{\tilde{e}'_\ell(1)}\,h_\ell^{(1)}(kr_<)\right]\left[h_\ell^{(1)}(kr_>) - \frac{\tilde{e}'_\ell(2)}{\tilde{s}'_\ell(2)}\,j_\ell(kr_>)\right] \tag{48}$$

where $k = |\omega|$ as before, and where we for simplicity have introduced the abbreviations $\tilde{s}_\ell(1) \equiv \tilde{s}_\ell(ka)$, $\tilde{s}_\ell(2) \equiv \tilde{s}_\ell(kb)$, etc. In region III, the Green functions are given by (11), with a replaced by b. We can thus find the two-point functions everywhere: in region I they

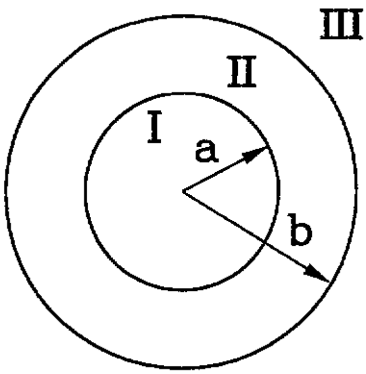

Figure 3. Geometry of the double spherical shell.

can be read off directly from (17) and (20)–(22), in region III they are given by (23)–(26) with $a \to b$, and in region II they are calculated by inserting (47) and (48) into (6)–(9). We shall not give the explicit two-point functions in region II here, but rather focus attention on the surface force density acting on one of the surfaces, say the force F_a on the inner surface. As usual, the force is calculated by taking the difference between the diagonal stress tensor components at $r = a-$ and $r = a+$. To simplify the notation, we write henceforth $\langle e_r^2(a\pm)\rangle$ instead of $\langle E_r(r)E_r(r')\rangle_{r'\to r=a\pm}$, etc. In order to exploit the simplification caused by the electromagnetic boundary conditions at the surface we use the two-point functions for the complete fields, rather than the 'effective' two-point functions. We then have

$$F_a = -\tfrac{1}{2}\langle E_r^2(a-)\rangle + \tfrac{1}{2}\langle H_\perp^2(a-)\rangle + \tfrac{1}{2}\langle E_r^2(a+)\rangle - \tfrac{1}{2}\langle H_\perp^2(a-)\rangle\,. \tag{49}$$

Here, the terms at $r = a-$ are found from (17) and (22). A brief calculation using (47) and (48) shows that the corresponding terms at $r = a+$ are

$$\langle E_r^2(a+)\rangle = \frac{-1}{\pi a^4}\int_0^{x_0}\frac{\mathrm{d}x}{x}\sum_{\ell=1}^{\infty}\frac{2\ell+1}{4\pi}\ell(\ell+1)\frac{e_\ell(x) - \frac{e_\ell'(bx/a)}{s_\ell'(bx/a)}s_\ell(x)}{e_\ell'(x) - \frac{e_\ell'(bx/a)}{s_\ell'(bx/a)}s_\ell'(x)} \tag{50}$$

$$\langle H_\perp^2(a+)\rangle = \frac{1}{\pi a^4}\int_0^{x_0}x\,\mathrm{d}x\sum_{\ell=1}^{\infty}\frac{2\ell+1}{4\pi} \times\left[\frac{e_\ell(x) - \frac{e_\ell'(bx/a)}{s_\ell'(bx/a)}s_\ell(x)}{e_\ell'(x) - \frac{e_\ell'(bx/a)}{s_\ell'(bx/a)}s_\ell'(x)} + \frac{e_\ell'(x) - \frac{e_\ell(bx/a)}{s_\ell(bx/a)}s_\ell'(x)}{e_\ell(x) - \frac{e_\ell(bx/a)}{s_\ell(bx/a)}s_\ell(x)}\right] \tag{51}$$

(note that primes mean differentiation with respect to the whole argument). If the outer shell recides to infinity, $b/a \to \infty$, then expressions (50) and (51) reduce to the single-shell expressions (23) and (26) evaluated at $r = a+$. Substitution into (49) yields

$$F_a = \frac{-1}{2\pi a^4}\int_0^{x_0}x\,\mathrm{d}x\sum_{\ell=1}^{\infty}\frac{2\ell+1}{4\pi}\left[\frac{s_\ell'(x)}{s_\ell(x)} + \frac{s_\ell''(x)}{s_\ell'(x)} + \frac{e_\ell'(x) - \frac{e_\ell(bx/a)}{s_\ell(bx/a)}s_\ell'(x)}{e_\ell(x) - \frac{e_\ell(bx/a)}{s_\ell(bx/a)}s_\ell(x)} + \frac{e_\ell''(x) - \frac{e_\ell'(bx/a)}{s_\ell'(bx/a)}s_\ell''(x)}{e_\ell'(x) - \frac{e_\ell'(bx/a)}{s_\ell'(bx/a)}s_\ell'(x)}\right]. \tag{52}$$

This is the generalization of (29) to the case of the double shell. No regularization procedure is to be imposed here: the eventual 'contact' term to be subtracted off would be the radial force across the fictitious surface $r = a$, if both shells were removed. Evidently such a term is zero, and so (52) gives the physical force directly.

We shall not evaluate the expression (52) in general. What we shall confine ourselves to is to rewrite it in a compact form which is convenient for further processing, and also to examine one important special case in detail, namely the one in which $a \to \infty$ at a fixed distance $d = b - a$ between the shells. The point is that we have herewith the opportunity to check the consistency of the formalism: the dominant term in F ought in this limiting case to approach the standard expression

$$F_{\mathrm{plate}} = \frac{\pi^2}{240}\frac{1}{d^4} \tag{53}$$

for the force between two flat parallel plates.

Let us first write the force such that the quantities

$$x = \hat{\omega} a \qquad y = \hat{\omega} b \tag{54}$$

appear as independent variables. Introducing as in previous works [20, 21], the quantity Q_ℓ and its derivatives,

$$\begin{aligned} Q_\ell &= s_\ell(x)e_\ell(y) - e_\ell(x)s_\ell(y) \\ Q_\ell''(x, y) &= s_\ell'(x)e_\ell'(y) - e_\ell'(x)s_\ell'(y) \end{aligned} \tag{55}$$

we can write the force in the following compact form:

$$F_a = \frac{-1}{2\pi a^4}\int_0^{x_0} x\,\mathrm{d}x \sum_{\ell=1}^{\infty}\frac{2\ell+1}{4\pi}\frac{\partial}{\partial x}\log\left[-s_\ell(x)s_\ell'(x)Q_\ell Q_\ell''(x, y)\right] \tag{56}$$

(the operator $\partial/\partial x$ is taken at a constant value of y). Next, in $Q_\ell Q_\ell''(x, y)$ we separate off the factor $e_\ell(x)e_\ell'(x)s_\ell(y)s_\ell'(y)$, in which the y-dependent terms do not contribute in view of the derivative in (56). We are left with

$$\begin{aligned} F_a = {} & \frac{-1}{2\pi a^4}\int_0^{x_0} x\,\mathrm{d}x \sum_{\ell=1}^{\infty}\frac{2\ell+1}{4\pi}\frac{\mathrm{d}}{\mathrm{d}x}\log[-s_\ell s_\ell' e_\ell e_\ell'] \\ & -\frac{1}{2\pi a^4}\int_0^{x_0} x\,\mathrm{d}x \sum_{\ell=1}^{\infty}\frac{2\ell+1}{4\pi}\frac{\partial}{\partial x}\log\left\{\left[1 - \frac{s_\ell(x)}{e_\ell(x)}\frac{e_\ell(y)}{s_\ell(y)}\right]\left[1 - \frac{s_\ell'(x)}{e_\ell'(x)}\frac{e_\ell'(y)}{s_\ell'(y)}\right]\right\} \end{aligned} \tag{57}$$

in which the first term is seen to be identical to the force (30) on a simple shell. In this way we have managed to separate out the most delicate part of the force. The remaining second term in (57) will, similar to the first term, be processed further by means of the Debye expansions (31)–(34). Because of the difference between x and y, there is a non-vanishing contribution to the force already from the $\mathcal{O}(\nu^0)$ terms in the expansions. Going one step further, up to $\mathcal{O}(1/\nu)$, we see that, since $\mathrm{d}\eta/\mathrm{d}z = 1/(tz)$,

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{s_\ell(x)}{e_\ell(x)} = \frac{1}{tz}\left(1 + \frac{2u_1}{\nu}\right)\mathrm{e}^{2\nu\eta} \tag{58}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{s_\ell'(x)}{e_\ell'(x)} = \frac{-1}{tz}\left(1 + \frac{2v_1}{\nu}\right)\mathrm{e}^{2\nu\eta}\,. \tag{59}$$

A parameter playing an important role here is

$$f = 2\left[\eta\left(\frac{y}{\nu}\right) - \eta\left(\frac{x}{\nu}\right)\right]. \tag{60}$$

So far, the radii a and b have been assumed to be arbitrary. From now on we consider the case when the curvatures are small. It means that the parameter ξ, in general, defined as $\xi = d/a = (b-a)/a$, is small compared to unity.

In this case we have, to $\mathcal{O}(1/\nu)$ in the Debye expansions,

$$\frac{s_\ell(x)}{e_\ell(x)}\frac{e_\ell(y)}{s_\ell(y)} = \frac{s'_\ell(x)}{e'_\ell(x)}\frac{e'_\ell(y)}{s'_\ell(y)} = \mathrm{e}^{-\nu f} \tag{61}$$

since terms containing $\xi z/\nu = \hat{\omega} d/\nu^2 = \mathcal{O}(1/\nu^2)$ are negligible. The result is the same if we were working on the level $\mathcal{O}(1/\nu^0)$ from the outset; the $\mathcal{O}(1/\nu)$ terms do not contribute. Since $\partial f/\partial x = -2/(tx)$, we then obtain

$$F_a = \frac{1}{4\pi a^4}\left(-\frac{x_0}{4\pi} + \frac{3}{64}\right) + \frac{1}{\pi^2 a^4}\sum_{\ell=1}^{\infty}\int_0^{x_0/\nu}\frac{\mathrm{d}z}{t(z)}\frac{\nu^3 \mathrm{e}^{-\nu f}}{1-\mathrm{e}^{-\nu f}} \tag{62}$$

where, for simplicity, we have assumed $x_0 > 1$ so that the expression (39) is an adequate approximation for (37). Now, $f = 2[\eta(bz/a) - \eta(z)]$.

Equation (62) gives the most general expression for the dispersive shell. Some numerical calculation is needed for further evaluation. (As η is a monotonically increasing function of its argument, f is positive, and the sum over ℓ always converges.) However, as mentioned above, an important impetus for the present analysis was to check the consistency with expression (53) for two plates. This is a non-dispersive result, and so we may achieve our aim without paying any attention to the dispersive terms at all in the following. That means we may ignore the first x_0 term in (62) and replace the upper limit x_0/ν in the integral by infinity. Thus

$$F_a^{\mathrm{nondisp}} = \frac{1}{4\pi a^4}\frac{3}{64} + \frac{1}{\pi^2 a^4}\int_0^{\infty}\frac{\mathrm{d}z}{t(z)}\sum_{\ell=1}^{\infty}\frac{\nu^3\mathrm{e}^{-\nu f}}{1-\mathrm{e}^{-\nu f}}\,. \tag{63}$$

We now introduce the geometric series

$$\frac{\mathrm{e}^{-\nu f}}{1-\mathrm{e}^{-\nu f}} = \mathrm{e}^{-\nu f} + \mathrm{e}^{-2\nu f} + \mathrm{e}^{-3\nu f} + \cdots \tag{64}$$

and make use of a result derived in [22] for the sum

$$\begin{aligned}\sum_{\ell=1}^{\infty}\nu^3\mathrm{e}^{-\nu f} &= \frac{27\mathrm{e}^{f} - 10 - 12\mathrm{e}^{-f} - 6\mathrm{e}^{-2f} + \mathrm{e}^{-3f}}{256\sinh^5(f/2)}\\ &= \frac{6}{f^4}[1+\mathcal{O}(f^4)]\end{aligned} \tag{65}$$

to obtain for the last term in (63), omitting $\mathcal{O}(f^4)$,

$$\frac{6}{\pi^2 a^4}\int_0^{\infty}\frac{\mathrm{d}z}{tf^4}\left(1+\frac{1}{2^4}+\frac{1}{3^4}+\cdots\right) = \frac{\pi^2}{15a^4}\int_0^{\infty}\frac{\mathrm{d}z}{tf^4}\,. \tag{66}$$

Inserting [22]

$$f = \frac{2\xi}{t}[1 - \tfrac{1}{2}\xi t^2 + \tfrac{1}{6}\xi^2(2t^2 + z^2t^4) + \mathcal{O}(\xi^3)] \tag{67}$$

we obtain for the term (66)

$$\frac{\pi^2}{240d^4}\int_0^{\infty} t^3\,\mathrm{d}z\big[1 + 2\xi t^2 - \tfrac{1}{6}\xi^2(8t^2 - 15t^4 + 4z^2t^4) + \mathcal{O}(\xi^3)\big]. \tag{68}$$

Altogether, carrying out the integrations over z, we finally find the non-dispersive force density to be

$$F_a^{\text{nondisp}} = \frac{1}{4\pi a^4}\frac{3}{64} + \frac{\pi^2}{240 d^4}\left[1 + \tfrac{4}{3}\xi + \tfrac{16}{45}\xi^2 + \mathcal{O}(\xi^3)\right]. \tag{69}$$

The dominant term here is seen to be precisely the flat plate result (equation (53)). Our consistency check does therefore give a satisfactory result.

Recall again that no regularization procedure has been used to obtain this result: the force F_a is the *total* force on the shell, consisting of an interior part F_{int} and an exterior part F_{ext}:

$$F_a = F_{\text{int}} + F_{\text{ext}}. \tag{70}$$

If we were to calculate F_{int} or F_{ext} separately, then regularization procedures would be needed (cf the following section). In the sum (70), however, the two infinities cancel out. All expressions are well defined.

One cautious remark, concerning the accuracy of the two terms in (69), ought to be made. Taking d to be of zeroth order we see, in view of the relation $1/a^4 = \xi^4/d^4$, that the first term in (69) is a fourth-order quantity. One might wish, therefore, to carry out the expansion in ξ two orders further in the second term, to obtain the contribution to the mutual force to the same formal accuracy as in the first term. It is to be observed, however, that the first term in (69) has its roots in the $\mathcal{O}(1/\nu^2)$ correction in (35), which in turn is obtained when the Debye expansions (31) and (32) are carried out to $\mathcal{O}(1/\nu^3)$. By comparison, the second term in (69) is obtained, as mentioned above, merely on the basis of the $\mathcal{O}(1/\nu)$ terms in the Debye expansions. The two terms in (69) therefore arise from different levels of approximations in the initial expansions, and their comparison should therefore be considered with some care. We abstain from carrying out the ξ expansion any further here.

4. Compact spherical ball

Consider finally the most general case among those studied in the present paper, namely a single spherical dispersive ball of radius a surrounded by a vacuum.

First, we give the expressions for the scalar Green functions:

$$r, r' < a: \quad F_\ell, G_\ell(r, r') = \mathrm{i}nk\, j_\ell(nkr_<)\left[h_\ell^{(1)}(nkr_>) - \tilde{A}_{F,G}(ka)\, j_\ell(nkr_>)\right] \tag{71}$$

$$r, r' > a: \quad F_\ell, G_\ell(r, r') = \mathrm{i}k\left[j_\ell(kr_<) - \tilde{B}_{F,G}(ka) h_\ell^{(1)}(kr_<)\right] h_\ell^{(1)}(kr_>). \tag{72}$$

Here $k = |\omega|$, and $n = n(\omega) = \sqrt{\varepsilon(\omega)}$ is the refractive index. Imposition of the electromagnetic boundary conditions across the surface $r = a$ yields for the coefficients

$$\tilde{A}_F(ka) = \frac{\tilde{e}_\ell(nka)\tilde{e}_\ell'(ka) - n\tilde{e}_\ell(ka)\tilde{e}_\ell'(nka)}{\tilde{s}_\ell(nka)\tilde{e}_\ell'(ka) - n\tilde{e}_\ell(ka)\tilde{s}_\ell'(nka)} \tag{73}$$

$$\tilde{A}_G(ka) = \frac{n\tilde{e}_\ell(nka)\tilde{e}_\ell'(ka) - \tilde{e}_\ell(ka)\tilde{e}_\ell'(nka)}{n\tilde{s}_\ell(nka)\tilde{e}_\ell'(ka) - \tilde{e}_\ell(ka)\tilde{s}_\ell'(nka)} \tag{74}$$

$$\tilde{B}_F(ka) = \frac{\tilde{s}_\ell(nka)\tilde{s}_\ell'(ka) - n\tilde{s}_\ell(ka)\tilde{s}_\ell'(nka)}{\tilde{s}_\ell(nka)\tilde{e}_\ell'(ka) - n\tilde{e}_\ell(ka)\tilde{s}_\ell'(nka)} \tag{75}$$

$$\tilde{B}_G(ka) = \frac{n\tilde{s}_\ell(nka)\tilde{s}_\ell'(ka) - \tilde{s}_\ell(ka)\tilde{s}_\ell'(nka)}{n\tilde{s}_\ell(nka)\tilde{e}_\ell'(ka) - \tilde{e}_\ell(ka)\tilde{s}_\ell'(nka)}. \tag{76}$$

Of main interest are the expressions for the two-point functions, in the limit $r' \to r$. They may be found by inserting (71)–(76) into the general expressions (6)–(9). It is, however, convenient here to subtract off the first terms in (71) and (72), i.e. the terms referring to homogeneous space regions—already from the beginning. That means that we will be working in terms of the 'effective' field products, being analogous to the expressions (42)–(45) already given for the single shell. A non-trivial point here in comparison to the cases studied earlier is that because of the presence of n in the arguments in the interior region, the two first terms in (71) and (72) do not become equal to each other at $r = a$.

It may be noted that our way of calculating the surface force on the shell is equivalent to using the complete two-point functions from the beginning, and thereafter subtracting off a contact term constructed according to the following prescription:

(i) the inner contact force be calculated as if the inner medium be filling all space;
(ii) the outer contact force similarly corresponding to the outer vacuum region be filling all space.

The effective electric field products in the interior region $r, r' < a$ are (for simplicity we again omit the subscripts $r' \to r$):

$$\langle E_r(r)E_r(r')\rangle_{\text{eff}} = \frac{-1}{\pi r^4}\int_0^{x_0}\frac{\mathrm{d}x}{n^3 x}\sum_{\ell=1}^{\infty}\frac{2\ell+1}{4\pi}\ell(\ell+1)A_G(x)s_\ell^2\left(\frac{nxr}{a}\right) \tag{77}$$

$$\langle E_\perp(r)E_\perp(r')\rangle_{\text{eff}} = \frac{1}{\pi r^2 a^2}\int_0^{x_0}\frac{x\,\mathrm{d}x}{n}\sum_{\ell=1}^{\infty}\frac{2\ell+1}{4\pi}\left\{A_F(x)s_\ell^2\left(\frac{nxr}{a}\right) - A_G(x)\left[s_\ell'\left(\frac{nxr}{a}\right)\right]^2\right\} \tag{78}$$

where $n = n(\mathrm{i}x)$. For reference purposes we give the frequency-rotated coefficients explicitly,

$$A_F(x) = \frac{e_\ell(nx)e_\ell'(x) - ne_\ell(x)e_\ell'(nx)}{s_\ell(nx)e_\ell'(x) - ne_\ell(x)s_\ell'(nx)} \tag{79}$$

$$A_G(x) = \frac{ne_\ell(nx)e_\ell'(x) - e_\ell(x)e_\ell'(nx)}{ns_\ell(nx)e_\ell'(x) - e_\ell(x)s_\ell'(nx)}\,. \tag{80}$$

In the exterior region we obtain similarly

$$\langle E_r(r)E_r(r')\rangle_{\text{eff}} = \frac{-1}{\pi r^4}\int_0^{x_0}\frac{\mathrm{d}x}{x}\sum_{\ell=1}^{\infty}\frac{2\ell+1}{4\pi}\ell(\ell+1)B_G(x)e_\ell^2\left(\frac{xr}{a}\right) \tag{81}$$

$$\langle E_\perp(r)E_\perp(r')\rangle_{\text{eff}} = \frac{1}{\pi r^2 a^2}\int_0^{x_0}x\,\mathrm{d}x\sum_{\ell=1}^{\infty}\frac{2\ell+1}{4\pi}\left\{B_F(x)e_\ell^2\left(\frac{xr}{a}\right) - B_G(x)\left[e_\ell'\left(\frac{xr}{a}\right)\right]^2\right\} \tag{82}$$

where

$$B_F(x) = \frac{s_\ell(nx)s_\ell'(x) - ns_\ell(x)s_\ell'(nx)}{s_\ell(nx)e_\ell'(x) - ne_\ell(x)s_\ell'(nx)} \tag{83}$$

$$B_G(x) = \frac{ns_\ell(nx)s_\ell'(x) - s_\ell(x)s_\ell'(nx)}{ns_\ell(nx)e_\ell'(x) - e_\ell(x)s_\ell'(nx)}\,. \tag{84}$$

One may note that all coefficients (73)–(76), as well as (79), (80), (83), (84), are real quantities. The relationship between the original and the frequency-rotated coefficients in the interior is $\tilde{A}_{F,G}(\mathrm{i}x) = (-1)^{\ell+1} A_{F,G}(x)$; the corresponding equation also holds for the $B_{F,G}$-coefficients in the exterior.

Because of the presence of n it is instructive, instead of calculating the surface force density F directly, to consider its interior part $F_{\rm int}$ and its exterior part $F_{\rm ext}$ separately. Starting with the interior, we have

$$F_{\rm int} = -\tfrac{1}{2}\hat{\varepsilon}\langle E_r^2(a-)\rangle_{\rm eff} + \tfrac{1}{2}\hat{\varepsilon}\langle E_\perp^2(a-)\rangle_{\rm eff} - \tfrac{1}{2}\langle H_r^2(a-)\rangle_{\rm eff} + \tfrac{1}{2}\langle H_\perp^2(a-)\rangle_{\rm eff} \tag{85}$$

cf (2). Inserting the expressions (77) and (78) along with their magnetic analogues we then obtain, when taking into account the governing equation for the Riccati–Bessel functions, the following compact form:

$$F_{\rm int} = \frac{1}{2\pi a^4}\int_0^{x_0} nx\,\mathrm{d}x \sum_{\ell=1}^{\infty} \frac{2\ell+1}{4\pi}\Lambda_\ell^{\rm int}\left[\log\left(\frac{s_\ell'(nx)}{s_\ell(nx)}\right)\right]'. \tag{86}$$

We have here introduced the symbol

$$\Lambda_\ell^{\rm int} = [A_F(x) + A_G(x)]s_\ell(nx)s_\ell'(nx) \tag{87}$$

(primes mean derivatives with respect to the whole argument, i.e. with respect to nx here). For the exterior force we obtain similarly

$$F_{\rm ext} = \frac{-1}{2\pi a^4}\int_0^{x_0} x\,\mathrm{d}x \sum_{\ell=1}^{\infty} \frac{2\ell+1}{4\pi}\Lambda_\ell^{\rm ext}\left[\log\left(\frac{-e_\ell'(x)}{e_\ell(x)}\right)\right]' \tag{88}$$

where

$$\Lambda_\ell^{\rm ext} = [B_F(x) + B_G(x)]e_\ell(x)e_\ell'(x)\,. \tag{89}$$

The total surface force density is $F = F_{\rm int} + F_{\rm ext}$.

We shall finally give approximate formulae for $F_{\rm int}$ and $F_{\rm ext}$ in limiting cases, again using the Debye expansions to $\mathcal{O}(1/\nu)$, ignoring dispersion. The derivative terms in (86) and (88) become to this order

$$\left[\log\left(\frac{s_\ell'(nx)}{s_\ell(nx)}\right)\right]' = -\frac{t^2(nz)}{\nu nz}\left[1 + \frac{3n^2z^2t^3(nz)}{2\nu}\right] \tag{90}$$

$$\left[\log\left(\frac{-e_\ell'(x)}{e_\ell(x)}\right)\right]' = -\frac{t^2(z)}{\nu z}\left[1 - \frac{3z^2t^3(z)}{2\nu}\right] \tag{91}$$

where $t^2(nz) = 1/(1+n^2z^2)$. These expressions hold for arbitrary values of n. The analogous expansions for (87) and (89) are for general n more unwieldy. We give the expressions only in two limiting cases: if $n \gg 1$, then to the same order

$$\Lambda_\ell^{\rm int}(n \gg 1) = \frac{1}{1+2z^2}\left[1 - \frac{2z^2t(z)}{\nu(1+2z^2)}\right] \tag{92}$$

$$\Lambda_\ell^{\rm ext}(n \gg 1) = \frac{t(z)t(nz)}{1 + (1+2z^2)t(z)t(nz)} + \frac{t^3(z)}{2\nu} \tag{93}$$

whereas for dilute media, $n \simeq 1$, we obtain

$$\Lambda_\ell^{\rm int}(n \simeq 1) = (n-1)t^2(z)\left[1 - \frac{3z^2t^3(z)}{2\nu}\right] \tag{94}$$

$$\Lambda_\ell^{\rm ext}(n \simeq 1) = \frac{(n-1)t^2(z)}{2}\left[1 + \frac{3z^2t^3(z)}{2\nu}\right]. \tag{95}$$

Consider now $F_{\rm int}^{\rm nondisp}$, when $n \gg 1$. The omission of dispersion means that n = const, $x_0 = \infty$ in (86). The last term in (90) is seen not to contribute to leading order in $1/n$, and so

$$F_{\rm int}^{\rm nondisp}(n \gg 1) = \frac{-1}{4\pi^2 a^4}\sum_{\ell=1}^{\infty}\nu^2\int_0^\infty \frac{t^2(nz)\,{\rm d}z}{1+2z^2}\left[1-\frac{2z^2t(z)}{\nu(1+2z^2)}\right]. \qquad (96)$$

Nor does the last integral here contribute to the leading order in $1/n$, and, including only the dominant contribution to the first integral, we obtain

$$F_{\rm int}^{\rm nondisp}(n \gg 1) = \frac{-1}{8\pi n a^4}\sum_{\ell=1}^{\infty}\nu^2. \qquad (97)$$

The divergent sum over ℓ needs regularization. The method that we shall use here is to exploit the analytic continuation of Riemann's zeta function. The only formula needed in practical calculation is

$$\sum_{\ell=0}^{\infty}\nu^s = (2^{-s}-1)\zeta(-s) \qquad (98)$$

with s an integer. The Riemann zeta-function method is simple and effective, and is used so often that it has become a standard method in quantum field theory. We shall briefly return to the legitimacy of this method in the concluding section. For the moment, let us simply put $s = 2$ in (98) to get

$$\sum_{\ell=1}^{\infty}\nu^2 = -\tfrac{1}{4} \qquad (99)$$

which means that the interior non-dispersive force to leading order in $1/n$ is

$$F_{\rm int}^{\rm nondisp}(n \gg 1) = \frac{1}{32\pi n a^4}. \qquad (100)$$

The result is the same as if we were working on the $\mathcal{O}(\nu^0)$ level from the outset. The force is repulsive, as we would expect. It is inversely proportional to n, and vanishes if $n \to \infty$. Again, this is a result that we would expect, since the fields must vanish inside a medium of infinite permittivity.

The exterior force in the same limit is found by inserting (93) and (91) into (88). Using (99), together with the substitutions

$$\sum_{\ell=1}^{\infty}\nu = -\tfrac{11}{24} \qquad \sum_{\ell=1}^{\infty}\nu^0 = -1 \qquad (101)$$

which are obtainable from (98), we find

$$F_{\rm ext}^{\rm nondisp}(n \gg 1) = \frac{1}{8\pi a^4}\left\{-\frac{1}{8\pi}\int_0^\infty \frac{t^3(z)t(nz)[4-11z^2t^3(z)]\,{\rm d}z}{1+(1+2z^2)t(z)t(nz)} - \frac{11}{36\pi}+\frac{3}{64}\right\}. \qquad (102)$$

Here the last two terms owe their existence to our use of the Debye expansions up to $\mathcal{O}(1/\nu)$; they would be absent on the $\mathcal{O}(\nu^0)$ level. If $n \to \infty$, the first term in (102) tends to zero. We thus see that in the limit of infinite permittivity it is the $\mathcal{O}(1/\nu)$ terms, rather than the $\mathcal{O}(\nu^0)$ terms, that are most important. The two last terms in (102) yield a compressive force.

The expression (102) in the limit $n \to \infty$ makes it possible to make an interesting consistency check of the formalism, by comparing it with the external Casimir energy $E_{\rm ext}$ calculated in [22] for a spherical ball whose medium satisfies the condition $\varepsilon\mu = 1$. In the limit $\varepsilon \to \infty$, i.e. $\mu \to 0$, the electromagnetic boundary conditions at the surface for such a medium ensure the exterior fields to be the same as outside the $n \to \infty$ ball considered in the present paper. From (3.22) in [22] we quote, omitting the time-splitting cut-off term,

$$E_{\mathrm{ext}} = \frac{1}{2a}\left(-\frac{11}{36\pi} + \frac{3}{64}\right). \tag{103}$$

In view of the general relation between surface force and energy

$$F_{\mathrm{ext}} = -(1/4\pi a^2) - \frac{\partial E_{\mathrm{ext}}}{\partial a}$$

we see that (102) and (103) are in perfect agreement, thus supporting the consistency of the theory.

If the medium is dilute we obtain by a similar calculation, inserting (94) and (95) into (86) and (88), the non-dispersive forces to first order in $(n-1)$,

$$F_{\mathrm{int}}^{\mathrm{nondisp}}(n \simeq 1) = \frac{n-1}{64\pi a^4} \tag{104}$$

$$F_{\mathrm{ext}}^{\mathrm{nondisp}}(n \simeq 1) = -\frac{n-1}{128\pi a^4}. \tag{105}$$

If $n > 0$ there is a repulsive force on the inside and a compressive force on the outside, the latter being only half as strong as the former. Adding (104) and (105) we thus see that the total non-dispersive surface force density is repulsive:

$$F^{\mathrm{nondisp}}(n \simeq 1) = \frac{n-1}{128\pi a^4}. \tag{106}$$

5. Conclusion and final remarks

Our calculation is based on a non-magnetic dispersive model for the medium. The temperature is $T = 0$. The adopted dispersion relation is as shown in figure 1; for simplicity it is taken to be a step function when viewed along the *imaginary* frequency axis $\hat{\omega}$.

The case of perfect conductivity corresponds in our model to $\varepsilon \to \infty$ for $\hat{\omega} \leqslant \omega_0$, while $\varepsilon = 1$ for $\hat{\omega} > \omega_0$. For a single, perfectly conducting shell our Green function calculation yields the general expression (37) for the surface force density. If $x_0 \equiv \omega_0 a \lesssim 1$, equation (39) is a useful approximation. The non-dispersive term in this expression is in agreement with Boyer [8], Milton *et al* [3], and others. The necessity of including also an attractive dispersive term, proportional to x_0, has been known for some years, since the paper of Candelas [4]. The dispersive term, proportional to x_0, has been found also in more recent model calculations [5, 6, 20]. The x_0 term makes it possible to revive the Casimir semiclassical electron model [9]: it corresponds to the specific value $x_0 = 0.397$ for the non-dimensional cut-off.

The perfectly conducting double spherical shell analysed in section 3 is the next step in complexity. Equation (57) gives the general surface force density on the inner shell. If the distance d between the shells is kept constant, while the radii a and b recede to infinity, then the dominant term in the non-dispersive part of the surface force is according to (69) in accordance with the standard result $\pi^2/240d^4$ for two parallel plates. This is a useful consistency check.

For the compact spherical ball considered in section 4 the inner/outer surface force densities are given by (86) and (88), respectively. Explicit expressions are given later in this section, in limiting cases. Some remarks are called for, as regards the legitimacy of the Riemann zeta-function regularization that we use: this method is simple and effective, leading generally to results that are in agreement with what one can obtain in other ways. The method has therefore been used frequently in previous works in the present field of research [23, 22, 20]. In this context we find it worthwhile to consider the following example in some detail, as it shows in a clear way the basic properties of the method: consider the force term given in dispersive theory by (3.17) in [20]. This term gives part of the surface force on a singular shell. Manipulate this term in the following way:

(i) eliminate the influence from dispersion by putting $x_0 = \infty$;
(ii) evaluate the remaining divergent sum over ℓ by the Riemann zeta-function method.

The result obtained is found to be identical to that obtained within non-dispersive theory, after the cut-off term involving the time-splitting parameter has been separated off [24]. The Riemann zeta function is generally effective to obtain the *non-dispersive* part of the force.

Another example of the same sort is provided by the expressions for the interior/exterior energies $E_{\rm int}$ and $E_{\rm ext}$ obtained in [22] for a singular shell; cf also (103) above. These expressions were calculated by other methods. However as remarked in [22], the most simple way of obtaining the non-dispersive expressions is precisely to make use of the Riemann zeta function.

One may still argue that the handling of infinite sums by means of (98) shows that the regularization is physically not properly understood. We agree with this attitude to some extent, although the divergent behaviour is not peculiar for Casimir theory but rather a common feature of quantum field theories in general.

Finally, we emphasize that the force calculations above were based upon the Minkowski energy–momentum tensor. (Under stationary conditions, this tensor is equivalent to the Abraham tensor.) One may wonder if not the electrostrictive contribution to the force [1] should have also been included in the force. We think that the point here is the specific way in which measurements are carried out. In most cases involving force measurements on dielectric media the electrostrictive contribution to the force does not play any role at all; cf the review article [25] on the energy–momentum tensor. However, if local stress measurements are carried out, then the electrostrictive contribution must also be taken into account. Electrostriction does not seem to have gained much attention in the past as far as the Casimir effect is concerned; the only work we are aware of is actually [26], although that work was limited to non-dispersive theory. For completeness, it would be desirable to calculate the electrostrictive contribution to the force along the lines of the present paper.

Appendix. Planar one-surface geometry

The reason for the angular momentum divergences in the non-regularized expressions encountered above is the *curvature* of the dielectric surface. Divergences of this sort should be absent if the surface were *planar*. Let us illustrate this point by considering the one-surface planar geometry of figure A1: for $z \geqslant 0$ there is a homogeneous non-magnetic medium, for $z < 0$ there is a vacuum. We shall calculate the surface force density $F_z = F$ on the boundary $z = 0$.

The governing equation for Γ is (4), as before. We introduce a transverse spatial Fourier transform,

$$\Gamma_{ik}(\boldsymbol{r}, \boldsymbol{r}', \omega) = \int \frac{\mathrm{d}\boldsymbol{q}}{(2\pi)^2}\, \mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot(\boldsymbol{r}-\boldsymbol{r}')}\, \Gamma_{ik}(z, z', \boldsymbol{q}, \omega) \tag{A1}$$

where $\boldsymbol{q}$ is the component of the total wavevector $\boldsymbol{k}$ transverse to the z-direction. When dealing with single Fourier components, we can without loss of generality choose $\boldsymbol{q}$ along

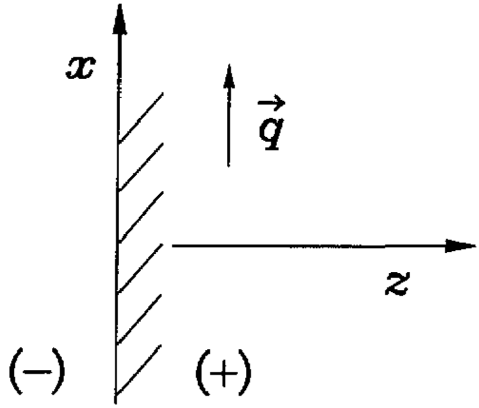

Figure A1. One planar dielectric surface, lying at $z = 0$.

the x-axis (figure A1). The diagonal components of $\Gamma_{ik}(z, z', \boldsymbol{q}, \omega)$ in the medium region $z \geqslant 0$ become [27, 28]

$$\Gamma_{xx} = -\frac{1}{\varepsilon}\delta(z-z') + \frac{\kappa}{2\varepsilon}\left(-\mathrm{e}^{-\kappa|z-z'|} + R^H \mathrm{e}^{-\kappa(z+z')}\right) \tag{A2}$$

$$\Gamma_{yy} = \frac{\omega^2}{2\kappa}\left(\mathrm{e}^{-\kappa|z-z'|} + R^E \mathrm{e}^{-\kappa(z+z')}\right) \tag{A3}$$

$$\Gamma_{zz} = -\frac{1}{\varepsilon}\delta(z-z') + \frac{q^2}{2\varepsilon\kappa}\left(\mathrm{e}^{-\kappa|z-z'|} + R^H \mathrm{e}^{-\kappa(z+z')}\right) \tag{A4}$$

where the electric and magnetic reflection coefficients are

$$R^E = \frac{\kappa - \kappa^{\mathrm{vac}}}{\kappa + \kappa^{\mathrm{vac}}} \qquad R^H = \frac{\kappa - \varepsilon\kappa^{\mathrm{vac}}}{\kappa + \varepsilon\kappa^{\mathrm{vac}}} \tag{A5}$$

with

$$\kappa(\omega) = \sqrt{q^2 - \varepsilon\omega^2} \qquad \kappa^{\mathrm{vac}}(\omega) = \sqrt{q^2 - \omega^2}. \tag{A6}$$

We take the limit $z' \to z$, but keep z' and z separated so that the delta functions above can be omitted; moreover we consider the boundary-induced field products only, i.e. the 'effective' products being analogous to the expressions (77), (78) and (81), (82) for the sphere. The effective products involve only the R^E, R^H terms in (A2)–(A4). After a complex frequency rotation we obtain, for $z, z' \geqslant 0$,

$$\langle E_z(z)E_z(z')\rangle_{\mathrm{eff}} = \frac{1}{4\pi^2}\int_0^\infty \frac{\hat{\omega}^3}{\varepsilon}\,\mathrm{d}\hat{\omega}\int_1^\infty \frac{p^2-1}{s}p\,\mathrm{d}p\frac{s-\varepsilon p}{s+\varepsilon p}\mathrm{e}^{-2s\hat{\omega}z} \tag{A7}$$

$$\langle E_\perp(z)E_\perp(z')\rangle_{\mathrm{eff}} = \frac{1}{4\pi^2}\int_0^\infty \frac{\hat{\omega}^3}{\varepsilon}\,\mathrm{d}\hat{\omega}\int_1^\infty \frac{p}{s}\,\mathrm{d}p\left[\frac{s^2(s-\varepsilon p)}{s+\varepsilon p} - \frac{\varepsilon(s-p)}{s+p}\right]\mathrm{e}^{-2s\hat{\omega}z} \tag{A8}$$

$$\langle H_z(z)H_z(z')\rangle_{\mathrm{eff}} = \frac{1}{4\pi^2}\int_0^\infty \hat{\omega}^3\,\mathrm{d}\hat{\omega}\int_1^\infty \frac{p^2-1}{s}p\,\mathrm{d}p\frac{s-p}{s+p}\mathrm{e}^{-2s\hat{\omega}z} \tag{A9}$$

$$\langle H_\perp(z)H_\perp(z')\rangle_{\mathrm{eff}} = \frac{1}{4\pi^2}\int_0^\infty \hat{\omega}^3\,\mathrm{d}\hat{\omega}\int_1^\infty \frac{p}{s}\,\mathrm{d}p\left[\frac{s^2(s-p)}{s+p} - \frac{\varepsilon(s-\varepsilon p)}{s+\varepsilon p}\right]\mathrm{e}^{-2s\hat{\omega}z} \tag{A10}$$

where $\varepsilon = \varepsilon(\mathrm{i}\hat{\omega})$, $\langle E_\perp^2\rangle = \langle E_x^2\rangle + \langle E_y^2\rangle$. We have here introduced the Lifshitz variables [29, 12] p and s, which can in terms of the variables above be written as $p = (1/\hat{\omega})\kappa^{\mathrm{vac}}(\mathrm{i}\hat{\omega})$,

$s = (1/\hat{\omega})\kappa(\mathrm{i}\hat{\omega})$, or explicitly

$$p = (1 + q^2/\hat{\omega}^2)^{1/2} \qquad s = (p^2 + \varepsilon - 1)^{1/2}. \tag{A11}$$

In the field products above, the limit $z' \to z$ is implicitly understood.

Consider now the surface force density $F(+)$ acting on the right-hand side of the surface. It is calculated to be

$$F(+) = -\tfrac{1}{2}\hat{\varepsilon}\langle E_z^2(+)\rangle_{\mathrm{eff}} + \tfrac{1}{2}\hat{\varepsilon}\langle E_\perp^2(+)\rangle_{\mathrm{eff}} - \tfrac{1}{2}\langle H_z^2(+)\rangle_{\mathrm{eff}} + \tfrac{1}{2}\langle H_\perp^2(+)\rangle_{\mathrm{eff}} = 0 \tag{A12}$$

when we insert (A7)–(A10) for $z \to 0$. The total surface force constructed as the sum of the forces on the two sides must therefore also vanish: $F = F(+) + F(-) = 0$.

There are thus no angular momentum divergences here. The example illustrates the general property that a curvature of the boundary is a necessary condition for this kind of divergences to occur.

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